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# On the structure of the free field equations

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**Abstract.** The structure of the Poincaré-covariant equations for free massive fields is analysed. It is assumed that the supplementary conditions follow from the field equations. By use of the notion of the commutant the general form of such equations is given. The necessary and sufficient conditions for their existence are found.

## 1. Introduction

The search for the higher-spin field equations is a long-standing problem in field theory. The relativistic theory of the free higher-spin fields first proposed by Fierz (1939) and Fierz and Pauli (1939) has been developed by Rarita and Schwinger (1941), Bargman and Wigner (1946), Gel'fand and Yaglom (1948), Gel'fand *et al* (1963), Gupta (1954), Moldauer and Case (1956), Fronsdal (1958), Umezawa (1952, 1956), Takahashi and Umezawa (1953, 1964), Aurilia and Umezawa (1967, 1969), Weinberg (1969), Hurley (1974), Chang (1967), Capri (1969), Shamaly and Capri (1971), Giesen (1975) and others.

The central problem lies in finding the free field equations from which the Klein–Gordon equation and the supplementary conditions follow. As is well known, the existence of such equations implies that the Lagrangian of the theory exists and an algebraically consistent canonical formalism can be constructed. Furthermore, a systematic investigation of the interacting systems can be undertaken. This last problem is of great importance because of the inconsistency exhibited for the equations with external field couplings (Velo and Zwanzinger (1969a, b), for recent literature see Cox (1976, 1977), Allock and Hall (1977, 1978)).

The purpose of this paper is to present the systematic description of the free massive field equations under the following conditions:

(A) the field equations are Poincaré covariant;

(B) the order of the partial differential field equations is equal to one or two (for tensor fields);

(C) the field equations fulfil the hermiticity condition;

(D) the Klein–Gordon equation and the supplementary conditions follow from the field equations.

This last requirement denotes that the field belongs to a definite representation (in general reducible) of the Poincaré group. Of course, most interesting is the unique spin case.

The search for the free field equations satisfying the above conditions was undertaken by a number of authors. Chang (1967), Capri (1969), Shamaly and Capri (1971) and Giesen (1975), for example, have already presented methods for finding some sorts of field equations. However, these methods can apply only to the restricted class of the representations of the Lorentz group.

Let us summarise the results obtained here. In § 2 the consequences of the Poincaré covariance are analysed. It is found that the operator $L(p)$, defined by the free field equation $L(p)\psi(p)=0$, belongs to the commutant $\boldsymbol{X}_p$ of the representation of the little group $R_p$ of the four-momentum $p_\mu$. Properties of this commutant are investigated with the help of the theorem due to Weyl (1939). The explicit form of the base elements of $\boldsymbol{X}_p$ for an arbitrary representation of the Lorentz group is given. The question arises how to determine the expansion coefficients of the operator $L(p)$ with respect to the base elements of $\boldsymbol{X}_p$ in order to fulfil the conditions B and C. The solution is presented in §§ 3 and 4. The requirement D is investigated in § 5 where the necessary and sufficient conditions for the existence of the operator $L(p)$ are given. Section 6 is devoted to giving an example illustrating the introduced formalism.

Summarising, the problem of construction of the free field equations is reduced to finding the nilpotent matrices or matrices for which a power is orthogonal to the physical spin space.

Finally, we remark that many results obtained here have already been found in the literature, partially or in a modified form.

## 2. Consequences of the relativistic covariance

Let the field $\phi(x)$ transform according to the law

$$\phi'(x)=D(\Lambda)\phi[\Lambda^{-1}(x-a)].$$

The finite-dimensional representation $D(\Lambda)$ of the Lorentz group can, in general, be reducible. The free field equations have the compact form

$$L(-\mathrm{i}\,\partial_\mu)\phi(x)=0$$

or in the momentum representation

$$L(p_\mu)\psi(p)=0. \tag{1}$$

Here $\psi(p)$ is the Fourier transform of $\phi(x)$. The condition of the relativistic covariance implies that $L(p)$ belongs to the set of the operators $X(p)$ defined by

$$X(\Lambda p)=D(\Lambda)X(p)D(\Lambda^{-1}). \tag{2}$$

Taking $\Lambda=R(p)\in R_p$, where $R_p$ is the little group of the four-momentum $p_\mu$, we see that

$$[X(p), D(R(p))]=0. \tag{2a}$$

This last equation denotes that the operators $X(p)$ form the commutant (or intertwining algebra) $\boldsymbol{X}_p$ of the representation $D(R(p))=D(\Lambda)\downarrow R_p$. Because in the $p^2>0$ case the little group $R_p\sim SO(3)\sim SU(2)$ is compact, Weyl's theorem (Weyl 1939, see also appendix 1) gives the complete characterisation of the commutant $\boldsymbol{X}_p$. In our case Weyl's theorem states that the set $\boldsymbol{X}_p$ forms the associative algebra which is the direct sum of the mutually orthogonal subalgebras $\boldsymbol{X}_p$ according to the decomposition $D(R)=\oplus_s N_s\mathscr{D}^s(R)$. Here $s$ is the spin, $\mathscr{D}^s$ denotes the irreducible representation of the SU(2) group and $N_s$ is the multiplicity of $\mathscr{D}^s$ in $D$. Furthermore, in each subalgebra

$\boldsymbol{X}_p^s$ there exists a basis $\{X_{ij}^s(p)\}$ with the multiplication law

$$X_{ij}^s(p)X_{kl}^{s'}(p)=\delta^{ss'}\delta_{jk}X_{il}^s(p). \tag{3}$$

Here the indices $i$, $k$, $j$, $l$ denote equivalent irreducible representations of the group $\mathrm{R}_p$ with fixed $s$. The operators $X_{ik}^s(p)$ intertwine these representations. Note that the subalgebras $\boldsymbol{X}_p^s$ are isomorphic to the algebra of $N_s \times N_s$ matrices and $\dim \boldsymbol{X}_p^s = N_s^2$. The irreducible subspaces of $D(\mathrm{R}_p)$ can be numbered by the triplet $[(A, B)\alpha]$ where $\alpha$ distinguishes the equivalent representations $D^{AB}$ of the Lorentz group ($A$ and $B$ are integer or half-integer) contained in $D$, namely

$$X_{ik}^s(p)\equiv X_{[(AB)\alpha],[(A'B')\alpha']}^s(p).$$

This follows from the fact that the representation $\mathscr{D}^s$ occurs in $D^{AB}$ with multiplicity one. Finally we note that the base elements $X_{ik}^s(p)$ are homogeneous with respect to $p_\mu$ with homogeneity degree equal to zero, i.e. they are dimensionless.

Analogously, we can consider the commutant $\boldsymbol{Y}$ of the representation $D(\Lambda)$. Weyl's theorem applies to this case because the finite-dimensional representations of the Lorentz group are completely reducible. From equation (2) it is obvious that the $p_\mu$-independent associative algebra $\boldsymbol{Y}$ belongs to the commutant $\boldsymbol{X}_p$. In $\boldsymbol{Y}$ we can choose the basis $\{Y_{\alpha\beta}^{(AB)}\}$ with the multiplication law

$$Y_{\alpha\beta}^{(AB)}Y_{\alpha'\beta'}^{(A'B')}=\delta^{AA'}\delta^{BB'}\delta_{\beta\alpha'}Y_{\alpha\beta'}^{(AB)}. \tag{4}$$

The operators $Y_{\alpha\beta}^{(AB)}$ intertwine the equivalent irreducible representations of the Lorentz group.

Let us consider the parity-invariant case. The invariance of the field equations under the parity transformation $\pi$,

$$\pi: \psi(p_0, \boldsymbol{p})\to \eta\psi(p_0, -\boldsymbol{p}),$$

of the field $\psi(p)$ implies that we must restrict ourselves to the subalgebra $\boldsymbol{X}_p^\pi$ of the $\boldsymbol{X}_p$ defined by

$$[\boldsymbol{X}_p^\pi, \pi]=0. \tag{5}$$

The base elements of the commutant $\boldsymbol{X}_p$ transform under the parity transformation as follows

$$X_{[(AB)\alpha],[(A'B')\alpha']}^s(p)\to X_{[(BA)\alpha],[(B'A')\alpha']}^s(p)=\eta X_{[(AB)\alpha],[(A'B')\alpha']}^s(p_0, -\boldsymbol{p})\eta^{-1}.$$

In the algebra $\boldsymbol{X}_p^\pi$ we can choose the basis defined by the relations

$$X_{[(AA)\alpha],[(A'A')\alpha']+}^s \doteqdot X_{[(AA)\alpha],[(A'A')\alpha']}^s, \tag{6a}$$

$$X_{[(AB)\alpha],[(A'B')\alpha']\pm}^s \doteqdot \tfrac{1}{2}[(X_{[(AB)\alpha],[(A'B')\alpha']}^s + X_{[(BA)\alpha],[(B'A')\alpha']}^s) \pm (X_{[(AB)\alpha],[(B'A')\alpha']}^s + X_{[(BA)\alpha],[(A'B')\alpha']}^s)] \qquad \text{if } A\neq B \text{ and } A'\neq B', \tag{6b}$$

$$X_{[(AB)\alpha],[(A'A')\alpha']+}^s \doteqdot (1/\sqrt{2})(X_{[(AB)\alpha],[(A'A')\alpha']}^s + X_{[(BA)\alpha],[(A'A')\alpha']}^s) \qquad \text{if } A\neq B, \tag{6c}$$

$$X_{[(AA)\alpha],[(A'B')\alpha']+}^s \doteqdot (1/\sqrt{2})(X_{[(AA)\alpha],[(A'B')\alpha']}^s + X_{[(AA)\alpha],[(B'A')\alpha']}^s) \qquad \text{if } A'\neq B'. \tag{6d}$$

From the multiplication rule (3) and the relations (6) we see that the algebra $\boldsymbol{X}_p^\pi$ is the direct sum of two mutually orthogonal subalgebras $\boldsymbol{X}^{\pi}_{p+}$ and $\boldsymbol{X}^{\pi}_{p-}$ with the multiplication law

$$
\begin{aligned}
X^{s_1}_{[(A_1B_1)\alpha_1],[(A_1'B_1')\alpha_1']\pm} & X^{s_2}_{[(A_2B_2)\alpha_2],[(A_2'B_2')\alpha_2']\pm} \\
& = \delta^{s_1s_2}\delta_{(A_1'B_1'),(A_2B_2)}\delta_{\alpha_1'\alpha_2}X^{s_1}_{[(A_1B_1)\alpha_1],[(A_2'B_2')\alpha_2']\pm}.
\end{aligned} \tag{7}
$$

It is easy to see that this decomposition of $\boldsymbol{X}^{\pi}_{p}$ is realised by the projectors $\Pi_{\pm}(p)=\frac{1}{2}[I\pm D(L_p)\eta D(L_p^{-1})]$ where $L_p$ is the Lorentz boost from $k_\mu=(m;0,0,0)$ to $p_\mu$. Note that

$$
X^{s}_{[(AB)\alpha],[(A'B')\alpha']\pm}=X^{s}_{[(BA)\alpha],[(A'B')\alpha']\pm}=X^{s}_{[(AB)\alpha],[(B'A')\alpha']\pm}.
$$

The same considerations apply to the subalgebra $\boldsymbol{Y}^{\pi}\subset\boldsymbol{X}^{\pi}_{p}$ of the commutant $\boldsymbol{Y}$, which is defined by

$$
[\boldsymbol{Y}^{\pi},\eta]=0.
$$

In $\boldsymbol{Y}^{\pi}$ we can choose the basis $\{Y^{\pi(AB)}_{\alpha\beta}\}$ as follows:

$$
Y^{\pi(AB)}_{\alpha\beta}\doteqdot Y^{(AB)}_{\alpha\beta}+Y^{(BA)}_{\alpha\beta}\qquad\text{if } A\neq B, \tag{8a}
$$

$$
Y^{\pi(AA)}_{\alpha\beta}\doteqdot Y^{(AA)}_{\alpha\beta}. \tag{8b}
$$

Finally we introduce the Dirac conjugation $\bar{\psi}\doteqdot\psi^{+}\eta$, $\overline{D^{AB}(\Lambda)}\doteqdot\eta^{-1}D^{AB+}(\Lambda)\eta=D^{BA}(\Lambda^{-1})$, which implies

$$
\bar{X}^{s}_{[(AB)\alpha],[(A'B')\alpha']}(p)=\eta^{-1}X^{s}_{[(AB)\alpha],[(A'B')\alpha']}(p)\eta=X^{s}_{[(B'A')\alpha'],[(BA)\alpha]}(p). \tag{9}
$$

As is well known, the matrix $\eta$ can be chosen to be Hermitian, i.e. $\eta^{+}=\eta$. Consequently an operator $X(p)\in\boldsymbol{X}_p$ is Hermitian in the space of states if $\bar{X}=X$.

The question arises how to construct explicitly the algebra $\boldsymbol{X}_p$ or $\boldsymbol{X}^{\pi}_{p}$. It is obvious that for the representations $D=(D^{\frac{1}{2}\frac{1}{2}}\otimes)^n$ or $(D^{\frac{1}{2}\frac{1}{2}}\otimes)^n\otimes(D^{\frac{1}{2}0}\oplus D^{0\frac{1}{2}})$ the base elements can be constructed from $p_\mu$, $\delta^{\nu}_{\mu}$, $g_{\mu\nu}$, $\epsilon_{\mu\nu\sigma\lambda}$ and (in the spinor case) $\gamma_\mu$. The parity-invariant operators cannot contain $\epsilon_{\mu\nu\sigma\lambda}(\gamma_5)$. We choose a set of linearly independent, appropriately symmetrised and antisymmetrised covariants constructed from $p_\mu$, $\delta^{\nu}_{\mu}$, $g_{\mu\nu}$, $\epsilon_{\mu\nu\sigma\lambda}$ and $\gamma_\mu$, which transform as $\psi\otimes\psi$. Then we project these operators on the subspaces with fixed $s$. The projectors can be constructed in standard fashion from the relativistic spin operator $\hat{s}^2\doteqdot-w^2/p^2$ where $w_\mu$ is the Pauli–Lubanski four-vector.

Now we derive the general formula for the matrix elements of the basis operator $X^{s}_{[(AB)\alpha],[(A'B')\alpha']}(p)$ in the so called SO(3) basis (spherical basis) of the arbitrary representation of the Lorentz group. We denote the base vectors of the representation by the kets $|ABsm\rangle_\alpha$ where $|A-B|\leqslant s\leqslant A+B$, $-s\leqslant m\leqslant s$. Firstly we note that

$$
\begin{aligned}
{}_{\alpha'}\langle A'B's'm'|X^{s}_{[(A_1B_1)\alpha_1],[(A_2B_2)\alpha_2]}(p)|A''B''s''m''\rangle_{\alpha''} & \\
= \delta_{\alpha'\alpha_1}\delta_{\alpha''\alpha_2}\delta_{A'A_1}\delta_{B'B_1}\delta_{A''A_2}\delta_{B''B_2} & \\
\times{}_{\alpha'}\langle A'B's'm'|X^{s}_{[(A'B')\alpha'],[(A''B'')\alpha'']}(p)|A''B''s''m''\rangle_{\alpha''}. &
\end{aligned}
$$

Furthermore, from the form of the Lorentz boost

$$
D(L_p)=\mathrm{e}^{-\mathrm{i}\phi J_3}\,\mathrm{e}^{-\mathrm{i}\sigma J_2}\,\mathrm{e}^{-\mathrm{i}\theta K_3}\,\mathrm{e}^{\mathrm{i}\sigma J_2}\,\mathrm{e}^{\mathrm{i}\phi J_3}
$$

where $\sin\theta=|\boldsymbol{p}|/\sqrt{p^2}$, $\cos\sigma=p_3/|\boldsymbol{p}|$, $\tan\phi=p_2/p_1$, and from the definition (2) we have

$$_{\alpha'}\langle A'B's'm'|X^{s}_{[(A'B')\alpha'],[(A''B'')\alpha'']}(p)|A''B''s''m''\rangle_{\alpha''}$$
$$= e^{i\phi(m''-m')}\ {}_{\alpha'}\langle A'B's'm'|\, e^{-i\sigma J_2}\, e^{-i\theta K_3} X^{s}_{[(A'B')\alpha'],[(A''B'')\alpha'']}(k)$$
$$\times e^{i\theta K_3}\, e^{i\sigma J_2}|A''B''s''m''\rangle_{\alpha''}.$$

Inserting between operators the identity $I = \Sigma\, |ABsm\rangle_{\alpha}\, {}_{\alpha}\langle ABsm|$ and taking into account the fact that the matrix element $_{\alpha'}\langle A'B'sm|X^{s}_{[(A'B')\alpha'],[(A''B'')\alpha'']}(k)|A''B''sm\rangle_{\alpha''}$ is equal to zero (if subspace $(A'B')$ or $(A''B'')$ does not contain a subspace with spin $s$) or one because $X^{s}_{[(A'B')\alpha'],[(A''B'')\alpha'']}(k)$ intertwine irreducible subspaces of the static ($\boldsymbol{p} = 0$) SO(3), we find that the non-zero matrix elements have the form

$$_{\alpha'}\langle A'B's'm'|X^{s}_{[(A'B')\alpha'],[(A''B'')\alpha'']}(p)|A''B''s''m''\rangle_{\alpha''}$$
$$= e^{i\phi(m''-m')} \sum_{m} d^{s'}_{m'm}(\sigma) D^{s's}_{m}(\theta) D^{ss''}_{m}(-\theta) d^{s''}_{mm''}(-\sigma).$$

Here $d^{s}_{mm'}(\sigma) \doteqdot \langle sm|\, e^{-i\sigma J_2}|sm'\rangle$ and $D^{ss'}_{m}(\theta) \doteqdot \langle sm|\, e^{-i\theta K_3}|s'm\rangle$. The matrix function $d^{s}_{mm'}(\sigma)$ can be simply calculated (see for example Werle (1966), equation (12.13)). Similarly $D^{ss'}_{m}(\theta)$ is given by the Clebsch–Gordan coefficients for SU(2):

$$D^{ss'}_{m}(\theta) = \sum_{ABaba'b'} \langle sm|ABab\rangle\langle ABab|\, e^{-i\theta K_3}|ABa'b'\rangle\langle ABa'b'|s'm\rangle$$
$$= \sum_{\substack{AB\\(a-b)}} e^{(b-a)\theta}\langle sm|ABab\rangle\langle s'm|ABab\rangle.$$

## 3. The degree with respect to $p_\mu$ of the elements of $\boldsymbol{X}_p$

The degree with respect to $p_\mu$ of the dimensionless operator $X(p)$ will be denoted below by $r(X(p))$. For example

$$r(\gamma_\mu) = r(\gamma_\mu\gamma^\nu) = r(\delta^{\gamma}_{\mu}) = 0, \qquad r(p^\mu/\sqrt{p^2}) = r(\gamma_5 + p\gamma/\sqrt{p^2}) = 1,$$
$$r\left(\frac{p^\mu}{\sqrt{p^2}}\delta^{\nu}_{\lambda} + \frac{p^\mu p^\nu p_\lambda}{(p^2)^{3/2}}\right) = 3, \qquad \text{etc.}$$

It is easy to see that the following rules hold:

$$r(X_aX_b) \leq r(X_a) + r(X_b), \tag{10a}$$

$$r(X_a + X_b) \leq \max\{r(X_a), r(X_b)\}, \tag{10b}$$

$$r(MX) = r(X), \tag{10c}$$

if $r(M) = 0$ and $M$ is invertible,

$$r(X(p')) = r(X(p)) \tag{10d}$$

where $p' = (p_0, R\boldsymbol{p})$, $R \in SO(3)$.

Let us consider a dimensionless operator $X(p) \in \boldsymbol{X}_p$. It can be expanded in the base $\{X^{s}_{[(AB)\alpha],[(A'B')\alpha']}(p)\}$ as follows:

$$X(p) = \sum_{[(AB)\alpha],[(A'B')\alpha']} \left(\sum_{s} \omega^{s}_{[(AB)\alpha],[(A'B')\alpha']} X^{s}_{[(AB)\alpha],[(A'B')\alpha']}(p)\right) \tag{11}$$

where the coefficients $\omega^{s}_{[(AB)\alpha],[(A'B')\alpha']}$ are $p_\mu$-independent. On the other hand we can rewrite $X(p)$ in the form

$$X(p)=\sum_{[(AB)\alpha],[(A'B')\alpha']} Y^{(AB)}_{\alpha\alpha} X(p) Y^{(A'B')}_{\alpha'\alpha'} \tag{11a}$$

where $Y^{(AB)}_{\alpha\alpha}$ and $Y^{(A'B')}_{\alpha'\alpha'}$ are the projectors on the representations $D^{AB}_{\alpha}$ and $D^{A'B'}_{\alpha'}$ respectively.

Now, we demand that

$$r(X(p))\leqslant l \tag{12}$$

where $l\geqslant 0$ is an integer. Because $r(Y)=0$ then from the equations (10$a$), (11), (11$a$) it follows that the condition (12) is satisfied if and only if the relations

$$r\Big(\sum_{s} \omega^{s}_{[(AB)\alpha],[(A'B')\alpha']} X^{s}_{[(AB)\alpha],[(A'B')\alpha']}(p)\Big)\leqslant l \tag{13}$$

hold for all $[(AB)\alpha]$ and $[(A'B')\alpha']$; $s$ varies from $\max(|A-B|,|A'-B'|)$ to $\min(A+B, A'+B')$. The solution of equation (13) is given by Kulesza and Rembieliński (1980) and has the following form:

$$\omega^{s}_{[(AB)\alpha],[(A'B')\alpha']}=\Big(\sum_{n} \lambda^{(n)}_{[(AB)\alpha],[(A'B')\alpha']}[s(s+1)]^{n}\Big)\omega^{s}_{(AB),(A'B')\,\min} \tag{14}$$

where $\lambda^{(n)}_{[(AB)\alpha],[(A'B')\alpha']}$ are arbitrary coefficients, $2\max(|A-A'|,|B-B'|)\leqslant l\leqslant 2\min(A+A', B+B')$, $2n=0, 2, 4, \ldots, [l-2\max(|A-A'|,|B-B'|)]$ and $\omega^{s}_{(AB),(A'B')\,\min}$ are given by

$$\omega^{s}_{(AB),(A'B')\,\min}=\prod_{i=0}^{N-1}[(s_{\mathrm{m}}-i)(s_{\mathrm{m}}-i+1)-s(s+1)]^{1/2} \times\prod_{k=0}^{M-1}[s(s+1)-(s_0+k)(s_0+k+1)]^{1/2} \tag{15a}$$

if $A\geqslant B$, $A'\geqslant B'$ or $A\leqslant B$, $A'\leqslant B'$,

$$\omega^{s}_{(AB),(A'B')\,\min}=\prod_{j=0}^{s_0-1}[s(s+1)-(s_0-j)(s_0-j-1)\prod_{i=0}^{N-1}[(s_{\mathrm{m}}-i)(s_{\mathrm{m}}-i+1)-s(s+1)]^{1/2} \times\prod_{k=0}^{M-1}[s(s+1)-(s_0+k)(s_0+k+1)]^{1/2} \tag{15b}$$

if $A>B$, $A'<B'$ or $A<B$, $A'>B'$ and both $A+B$ and $A'+B'$ are integer,

$$\omega^{s}_{(AB),(A'B')\,\min}=(2s+1)\prod_{j=0}^{s_0-3/2}[s(s+1)-(s_0-j)(s_0-j-1)] \times\prod_{i=0}^{N-1}[(s_{\mathrm{m}}-i)(s_{\mathrm{m}}-i+1)-s(s+1)]^{1/2} \times\prod_{k=0}^{M-1}[s(s+1)-(s_0+k)(s_0+k+1)]^{1/2} \tag{15c}$$

if $A>B$, $A'<B'$ or $A<B$, $A'>B'$ and both $A+B$ and $A'+B'$ are half odd integer. Here $N=|(A+B)-(A'+B')|$, $s_0=\min(|A-B|,|A'-B'|)$, $s_{\mathrm{m}}=\max(A+B, A'+B')$.

Finally, we note that from the above solution it follows that in equation (13)

(a) the maximal and minimal degrees of $l$ are equal respectively to $l_{\max} = 2\min(A+A', B+B')$ and $l_{\min} = 2\max(|A-A'|, |B-B'|)$;

(b) for fixed $(AB)$ and $(A'B')$, $l$ is only even or only odd;

(c) $X^{s}_{[(AB)\alpha],[(A'B')\alpha']}$ does not contain the terms with degree less than $l_{\min}$ (see also equation (11) in Kulesza and Rembieliński (1980)).

## 4. The free field equations

In this section the results of the preceding two sections are applied to the determination of the structure of the free field equations consistently with the conditions A, B and C of § 1. According to the condition B, the operator $L(p)$ defined by the field equation (1) has the form

$$L(p) = \sum_{k=0}^{q} L_k(p) \tag{16}$$

where $r(L_k/(\sqrt{p^2})^k) = k$ and $q = 1$ or 2. Because $L_k(p)$ is homogeneous of degree $k$ in $p$ then Lorentz covariance implies that each $L_k/(\sqrt{p^2})^k$ belongs to the commutant $\boldsymbol{X}_p$. Moreover, the parity invariance implies that $L_k/(\sqrt{p^2})^k$ belongs to the subalgebra $\boldsymbol{X}^{\pi}_p$ rather. In particular, the $p_\mu$-independent operator $L_0$ is contained in the algebra $\boldsymbol{Y}^{\pi} \subset \boldsymbol{X}^{\pi}_p$. Consequently, the operators $L_k$ take the form

$$L_k(p) = (\sqrt{p^2})^k \sum_{\substack{[(AB)\alpha],[(A'B')\alpha'] \\ A\geqslant B, A'\geqslant B'}} \sum_s [\omega^{s}_{k[(AB)\alpha],[(A'B')\alpha']+} X^{s}_{[(AB)\alpha],[(A'B')\alpha']+}(p)$$
$$+ \omega^{s}_{k[(AB)\alpha],[(A'B')\alpha']-} X^{s}_{[(AB)\alpha],[(A'B')\alpha']-}(p)] \qquad \text{for } k \neq 0, \tag{17a}$$

$$L_0 = \sum_{\substack{(AB),\alpha,\beta \\ A\geqslant B}} v^{(AB)}_{\alpha\beta} Y^{\pi(AB)}_{\alpha\beta} \qquad \text{for } k = 0. \tag{17b}$$

The coefficients $\omega^{s}_{k[(AB)\alpha],[(A'B')\alpha']\pm}$ and $v^{(AB)}_{\alpha\beta}$ will be determined later. Now we concentrate our attention on the hermiticity condition C. As is mentioned in § 2 (see equation (9) and below), the hermiticity of the operator $X$ in the Hilbert space is equivalent to the condition $\bar{X} = X$. From the equations (6), (8), (9) we have

$$\bar{X}^{s}_{[(AB)\alpha],[(A'B')\alpha']\pm}(p) = X^{s}_{[(A'B')\alpha'],[(AB)\alpha]\pm}(p)$$

and

$$\bar{Y}^{\pi(AB)}_{\alpha\beta} = Y^{\pi(AB)}_{\beta\alpha}.$$

Therefore from the hermiticity of $L$, i.e. from the conditions $\bar{L}_k = L_k$, we obtain

$$\omega^{s*}_{k[(AB)\alpha],[(A'B')\alpha']\pm} = \omega^{s}_{k[(A'B')\alpha'],[(AB)\alpha]\pm} \qquad \text{for } k \neq 0 \tag{18a}$$

and

$$v^{(AB)*}_{\alpha\beta} = v^{(AB)}_{\beta\alpha}. \tag{18b}$$

Note that $L_0$ can be diagonalised by the inner automorphism in the algebra $\boldsymbol{Y}^{\pi}$. By the formula (10c) this automorphism does not change the degree of $L_k/(\sqrt{p^2})^k$, and induces only the point transformation of the field $\psi$. Thus we can choose $L_0$ in the diagonal form

$$L_0=\sum_{\substack{[(AB)\alpha]\\ A\geqslant B}} v_\alpha^{(AB)} Y_{\alpha\alpha}^{\pi(AB)} \tag{19}$$

where the non-zero coefficients† $v_\alpha^{(AB)}$ are real.

Let us consider the consequences of the requirement B. For this purpose we denote by $\{AB\}$ the set $\{|A-B|, |A-B|+1, \ldots, (A+B)\}$ and rewrite equation (17$a$) in the following form

$$\begin{aligned} L_k(p)=(\sqrt{p^2})^k \sum_{\substack{[(AB)\alpha],[(A'B')\alpha']\\ A\geqslant B, A'\geqslant B'}} 2^{-t}\sum_s [(&\overset{s}{\omega}_{k[(AB)\alpha],[(A'B')\alpha']+}+\overset{s}{\omega}_{k[(AB)\alpha],[(A'B')\alpha']-})\\ &\times(X^s_{[(AB)\alpha],[(A'B')\alpha']}+X^s_{[(BA)\alpha],[(B'A')\alpha']})\\ &+(\overset{s}{\omega}_{k[(AB)\alpha],[(A'B')\alpha']+}-\overset{s}{\omega}_{k[(AB)\alpha],[(A'B')\alpha']-})\\ &\times(X^s_{[(AB)\alpha],[(B'A')\alpha']}+X^s_{[(BA)\alpha],[(A'B')\alpha']})]. \end{aligned} \tag{20}$$

Here the definitions (6$a$)–(6$d$) are used, $t=1$ if $A\neq B$ and $A'\neq B'$, $t=\frac{3}{2}$ if $A=B$ and $A'\neq B'$ or $A\neq B$ and $A'=B'$, $t=2$ if $A=B$ and‡ $A'=B'$. Now from equations (14) and (15) we determine the coefficients $\overset{s}{\omega}_{k[(AB)\alpha],[(A'B')\alpha']\pm}$. We start from the case $k=1$. As is mentioned in § 3, the minimal degree $l_{\min}$ of the operator

$$\sum_s \overset{s}{\omega}_{[(AB)\alpha],[(A'B')\alpha']}X^s_{[(AB)\alpha],[(A'B')\alpha']}(p)$$

is equal to $2\max(|A-A'|, |B-B'|)$. Moreover $l$ is only odd or only even. Thus in equation (20) the coefficients $\overset{s}{\omega}_{1[(AB)\alpha],[(A'B')\alpha']\pm}$ and the intertwining operators $X^s_{[(AB)\alpha],[(A'B')\alpha']}$ must fulfil the following conditions:

$$2\max(|A-A'|, |B-B'|)=1 \qquad \text{and for } A\neq B,\ A'\neq B',$$

$$\overset{s}{\omega}_{1[(AB)\alpha],[(A'B')\alpha']+}=\overset{s}{\omega}_{1[(AB)\alpha],[(A'B')\alpha']-}$$

or

$$2\max(|A-B'|, |A'-B|)=1 \qquad \text{and for } A\neq B,\ A'\neq B',$$

$$\overset{s}{\omega}_{1[(AB)\alpha],[(A'B')\alpha']+}=-\overset{s}{\omega}_{1[(AB)\alpha],[(A'B')\alpha']-}$$

(for $A=B$ or $A'=B'$ we have $\overset{s}{\omega}_{1[\ldots],[\ldots]-}=0$) where $A\geqslant B$, $A'\geqslant B'$. Therefore from equations (14) and (15) we have§

$$\overset{s}{\omega}_{1[(AB)\alpha],[(A'B')\alpha']+}=\lambda_{1[(AB)\alpha],[(A'B')\alpha']}(|s(s+1)-\tau(\tau+1)|)^{1/2} \tag{21$a$}$$

and for $A\neq B$, $A'\neq B'$,

$$\overset{s}{\omega}_{1[(AB)\alpha],[(A'B')\alpha']+}=\overset{s}{\omega}_{1[(AB)\alpha],[(A'B')\alpha']-}$$

if $\mathrm{card}[(\{AB\}\cup\{A'B'\})-(\{AB\}\cap\{A'B'\})]=1$ and where $\tau\in[(\{AB\}\cup\{A'B'\})-(\{AB\}\cap\{A'B'\})]$,

$$\overset{s}{\omega}_{1[(A,A-\frac{1}{2})\alpha],[(A,A-\frac{1}{2})\alpha']+}=-\overset{s}{\omega}_{1[(A,A-\frac{1}{2})\alpha],[(A,A-\frac{1}{2})\alpha']-}=\lambda_{1[(A,A-\frac{1}{2})\alpha],[(A,A-\frac{1}{2})\alpha']}(2s+1). \tag{21$b$}$$

The other coefficients $\overset{s}{\omega}_{1[(AB)\alpha],[(A'B')\alpha']\pm}$ vanish.

† The invertibility of the matrix $L_0$ guarantees that the massless modes are absent.
‡ Note that in the last three cases (for $A=B$ or $A'=B'$) the coefficients $\overset{s}{\omega}_{k[(AB)\alpha],[(A'B')\alpha']-}$ do not appear in equation (20) (see equations (6$a$)–(6$d$)).
§ We use standard set-theoretical notation: $\cup$—sum, $\cap$—product, card—cardinal number.

In similar fashion we obtain for the case $k=2$ the following formulae

$$\omega^{s}_{2[(AA)\alpha],[(AA)\alpha']+} = \lambda_{0[(AA)\alpha],[(AA)\alpha']} + s(s+1)\lambda_{2[(AA)\alpha],[(AA)\alpha']},$$

$$\omega^{s}_{2[(AA)\alpha],[(AA)\alpha']-} = 0, \tag{22a}$$

$$\omega^{s}_{2[(A,A-1)\alpha],[(A,A-1)\alpha']\pm} = \lambda_{0[(A,A-1)\alpha],[(A,A-1)\alpha']\pm} + s(s+1)\lambda_{2[(A,A-1)\alpha],[(A,A-1)\alpha']}, \tag{22b}$$

$$\omega^{s}_{2[(A,A-j)\alpha],[(A,A-j)\alpha']+} = \omega^{s}_{2[(A,A-j)\alpha],[(A,A-j)\alpha']-}$$

$$= \lambda_{0[(A,A-j)\alpha],[(A,A-j)\alpha']} + s(s+1)\lambda_{2[(A,A-j)\alpha],[(A,A-j)\alpha']}, \tag{22c}$$

for $j=2,3,\ldots,A$; if $\mathrm{card}[(\{AB\}\cup\{A'B'\})-(\{AB\}\cap\{A'B'\})]=2$ then

$$\omega^{s}_{2[(AB)\alpha],[(A'B')\alpha']\pm} = \lambda_{2[(AB)\alpha],[(A'B')\alpha']\pm}(|[s(s+1)-\tau_1(\tau_1+1)]\,.\,[s(s+1)-\tau_2(\tau_2+1)]|)^{1/2} \tag{22d}$$

where $\tau_1, \tau_2 \in [(\{AB\}\cup\{A'B'\})-(\{AB\}\cap\{A'B'\})]$ and for $B=A-i$, $i=3,4,\ldots,A$ we have $\lambda_{2[(AB)\alpha],[(A'B')\alpha']+} = \lambda_{2[(AB)\alpha],[(A'B')\alpha']-}$ while for $B=A$, $B=A-1$ or $B=A-2$ the constants $\lambda_{2[(AB)\alpha],[(A'B')\alpha']+}$ and $\lambda_{2[(AB)\alpha],[(A'B')\alpha']-}$ can differ in general. The other coefficients $\omega^{s}_{2[\ldots],[\ldots]\pm}$ vanish. Note that the hermiticity condition (18$a$) becomes

$$\lambda^{*}_{a[(AB)\alpha],[(A'B')\alpha']\pm} = \lambda_{a[(A'B')\alpha'],[(AB)\alpha]\pm} \tag{23}$$

for $a=0,1,2$.

## 5. Existence conditions for the free field equations

In this section we will analyse the necessary and sufficient conditions for existence of the operator $L(p)$ satisfying the requirement D in § 1. These conditions determine the coefficients $\omega^{s}_{k[(AB)\alpha],[(A'B')\alpha']\pm}$ and $v^{(AB)}_{\alpha}$ definitively.

Let us formulate the problem. We demand that the supplementary conditions

$$(I-\Pi(p))\psi(p)=0 \tag{24}$$

and the Klein–Gordon equation

$$(p^2-m^2)\psi(p)=0 \tag{25}$$

follow from the field equation (1). Here the operator $\Pi(p)\in \boldsymbol{X}_p$ projects on the (in general reducible) sub-representation space of the representation of the Poincaré group. This means that the field $\psi(p)$ contains a definite mixture of spins. In particular, we can demand the uniqueness of the spin of the field.

If the equations (24) and (25) follow from the field equation (1), then the operators $M(p)$ and $D(p)$ defined by the relations

$$M(p)L(p)=I-\Pi(p) \tag{26}$$

and

$$D(p)L(p)=(p^2-m^2)I \tag{27}$$

should exist. Writing $M(p)$ and $D(p)$ in the form $M(p)=\Sigma_r\,(\sqrt{p^2})^r M_r(p)$ and $D(p)=\Sigma_j\,(\sqrt{p^2})^j D_j(p)$, where $M_r(p)$ and $D_j(p)$ are dimensionless with respect to $p_\mu$, and using equations (16), (26) and (27), we obtain the following necessary and sufficient conditions (for details see appendix 2). $M(p)$ exists iff for a natural number $n$ the following equations hold

$$(I-\Pi)\sum_{r=0}^{[(n+1)/2]}\sum_{\substack{\text{all symmetrised}\\ \text{permutations}}}(-L_0^{-1}L_2)^r(-L_0^{-1}L_1)^{n+1-2r}=0, \tag{28a}$$

$$(I-\Pi)\left(\sum_{r=0}^{[n/2]}\sum_{\substack{\text{all symmetrised}\\ \text{permutations}}}(-L_0^{-1}L_2)^r(-L_0^{-1}L_1)^{n-2r}\right)(L_0^{-1}L_2)=0. \tag{28b}$$

$D(p)$ exists iff for a natural number $m$ the following equations hold

$$p^2\sum_{r=0}^{[(m-1)/2]}\sum_{\substack{\text{all symmetrised}\\ \text{permutations}}}(-L_0^{-1}L_2)^r(-L_0^{-1}L_1)^{m-1-2r}$$
$$-m^2\sum_{l=0}^{[(m+1)/2]}\sum_{\substack{\text{all symmetrised}\\ \text{permutations}}}(-L_0^{-1}L_2)^l(-L_0^{-1}L_1)^{m+1-2l}=0, \tag{29a}$$

$$\left(p^2\sum_{r=0}^{[(m-2)/2]}\sum_{\substack{\text{all symmetrised}\\ \text{permutations}}}(-L_0^{-1}L_2)^r(-L_0^{-1}L_1)^{m-2-2r}\right.$$
$$\left.-m^2\sum_{l=0}^{[m/2]}\sum_{\substack{\text{all symmetrised}\\ \text{permutations}}}(-L_0^{-1}L_2)^l(-L_0^{-1}L_1)^{m-2l}\right)(L_0^{-1}L_2)=0. \tag{29b}$$

Here the symbol $[f]$ denotes the Ent function, while the symmetrised sum $\Sigma_{\text{all symm. permut.}}X_2^aX_1^b$ is defined as the sum of all different permutations of the product

$$\underbrace{X_2\ldots X_2}_{a}\,\underbrace{X_1\ldots X_1}_{b}$$

with the coefficients equal to one. For example,

$$\sum_{\text{all symm. permut.}}X_2^1X_1^2=X_2X_1^2+X_1X_2X_1+X_1^2X_2.$$

If the equations (28) and (29) are satisfied then the operators $M(p)$ and $D(p)$ have the form

$$M(p)=(I-\Pi)\left(\sum_{k=0}^{n}\sum_{r=0}^{[k/2]}\sum_{\substack{\text{all symmetrised}\\ \text{permutations}}}(-L_0^{-1}L_2)^r(-L_0^{-1}L_1)^{k-2r}\right)L_0^{-1}, \tag{30}$$

$$D(p)=\left(p^2\sum_{k=0}^{m-2}\sum_{r=0}^{[k/2]}\sum_{\substack{\text{all symmetrised}\\ \text{permutations}}}(-L_0^{-1}L_2)^r(-L_0^{-1}L_1)^{k-2r}\right.$$
$$\left.-m^2\sum_{k=0}^{m}\sum_{l=0}^{[k/2]}\sum_{\substack{\text{all symmetrised}\\ \text{permutations}}}(-L_0^{-1}L_2)^l(-L_0^{-1}L_1)^{k-2l}\right)L_0^{-1}. \tag{31}$$

Note that for $L_2(p)=0$ the equations (28)–(31) simplify considerably, and instead of the above relations we have

$$(I-\Pi)(L_0^{-1}L_1)^{n+1}=0, \tag{32}$$

$$[p^2-m^2(L_0^{-1}L_1)^2](L_0^{-1}L_1)^{m-1}=0, \tag{33}$$

$$M(p)=(I-\Pi)\left(\sum_{k=0}^{n}(-L_0^{-1}L_1)^k\right)L_0^{-1}, \tag{34}$$

$$D(p)=[p^2-m^2(L_0^{-1}L_1)^2]\left(\sum_{k=0}^{m-2}(-L_0^{-1}L_1)^k\right)L_0^{-1}, \qquad \text{for } m>1, \tag{35a}$$

and for $m=0, 1$

$$D(p)=-m^2L_0^{-1}\sum_{k=0}^{m}(-L_0^{-1}L_1)^k. \tag{35b}$$

Similarly for $L_1=0$ the equations (28)–(31) have the form

$$(I-\Pi)(L_0^{-1}L_2)^{n/2+1}=0, \tag{36}$$

$$[p^2+m^2(L_0^{-1}L_2)](L_0^{-1}L_2)^{m/2}=0, \tag{37}$$

$$M(p)=(I-\Pi)\left(\sum_{k=0}^{n/2}(-L_0^{-1}L_2)^k\right)L_0^{-1}, \tag{38}$$

$$D(p)=[p^2+m^2(L_0^{-1}L_2)]\left(\sum_{k=0}^{m/2-1}(-L_0^{-1}L_2)^k\right)L_0^{-1}. \tag{39}$$

Here $n$ and $m$ are even. We note that from the orthogonality of the subalgebras $\boldsymbol{X}^{\pi s}_{p\pm}$ it follows that the conditions (28)–(29) (or the simplified versions (32)–(33) and (36)–(37)) are satisfied in each subalgebra $\boldsymbol{X}^{\pi s}_{p\pm}$ separately. Furthermore these relations are in fact the matrix equations. This follows from the isomorphism mentioned in § 2 between the subalgebras $\boldsymbol{X}^{\pi s}_{p\pm}$ and the algebras of $N_{s\pm}\times N_{s\pm}$ matrices, where $N_{s\pm}=(\dim \boldsymbol{X}^{\pi s}_{p\pm})^{1/2}$. The matrix elements of $L_k$ are given by equations (17), (19) and (21)–(23). Note that the matrices representing† $L_k$, $M$ and $D$ are Hermitian. It is remarkable that the equations (32) and (36) define the singular matrices $(L_0^{-1}L_1/\sqrt{p^2})^s_\pm$ and $(L_0^{-1}L_2/p^2)^s_\pm$ whose $(n+1)$ and $(n/2+1)$ powers respectively are orthogonal to the projector $(I-\Pi)^s_\pm$. The solution of this problem is known at least in the Jordan basis (see for example Gantmacher (1959)).

Now let us consider the special case when $\psi(p)$ has the unique spin $\sigma$. This means that for the tensor representations the projector $\Pi(p)$ projects on the irreducible subspace with spin $\sigma$, while for the spinor ones it projects on the direct sum of the particle and antiparticle subspaces with spin $\sigma$, i.e. $\Pi=\Pi^\sigma_{j+}+\Pi^\sigma_{j-}$. It is not difficult to prove that in the unique spin case the existence of $M(p)$ implies the existence of $D(p)$. For example, in the fermion case the relation (32) implies equation (33) for $m>n+1$. In the following we restrict ourselves for simplicity to the case $L_2=0$. Thus we can investigate the equation (32) only. We will denote by $\Pi^s_\pm(p)$ the projectors on whole parity-invariant subspaces with spin $s$, namely $\Pi^s_\pm(p)=\Sigma_{[(AB)\alpha]}X^s_{[(AB)\alpha],[(AB)\alpha]\pm}(p)$. The projector $\Pi(p)=\Pi^\sigma_{j+}+\Pi^\sigma_{j-}$, where $\Pi^\sigma_{j\pm}(p)$ project on the $j$th irreducible subspace $(\pm)$ with spin $\sigma$. Then equation (32) becomes

$$\left(L^{s^{-1}}_{0\pm}\frac{L^s_{1\pm}}{\sqrt{p^2}}\right)^{n+1}=0 \tag{40a}$$

† In the basis $\{X^s_{[(AB)\alpha],[(A'B')\alpha']\pm}\}$ the coefficients $\omega^s_{k[(AB)\alpha],[(A'B')\alpha']\pm}$ and $v^{(AB)}_\alpha$ are the matrix elements of $L_k/(\sqrt{p^2})^k$. In the following the term 'matrix $L_k/(\sqrt{p^2})^k$' denotes the matrix build from $\omega^s_{k[\ldots],[\ldots]\pm}$ or $v^{(AB)}_\alpha$.

for every $s \neq \sigma$ and

$$(\Pi^{\sigma}_{\pm} - \Pi^{\sigma}_{j\pm})\left(L^{\sigma^{-1}}_{0\pm}\frac{L^{\sigma}_{1\pm}}{\sqrt{p^2}}\right)^{n+1} = 0 \tag{40b}$$

for $s = \sigma$. Here $L^{s}_{k\pm} \doteqdot \Pi^{s}_{\pm}L_k = L_k\Pi^{s}_{\pm}$. Thus equations (40*b*) are satisfied by the matrices whose $(n+1)$th power is orthogonal to the projectors $\Pi^{\sigma}_{\pm} - \Pi^{\sigma}_{j\pm}$. The matrix elements

$$\frac{1}{\sqrt{p^2}} L^{s}_{1[(AB)\alpha],[(A'B')\alpha']\pm} = \omega^{s}_{1[(AB)\alpha],[(A'B')\alpha']\pm}$$

and

$$L^{s}_{0[(AB)\alpha],[(A'B')\alpha']\pm} = \delta_{(AB),(A'B')}\delta_{\alpha\alpha'}v^{(AB)}_{\alpha}$$

given by equations (19), (21) and (23) are determined definitely by the equations (40*a*, *b*). Note that the matrices corresponding to the different values of $s$ are interdependent in view of equations (19)–(23).

## 6. The example

Now we give the standard example illustrating the introduced formalism. Let us consider the Rarita–Schwinger theory for the spin-$\frac{3}{2}$ field. The representation $D$ is chosen as $D^{\frac{1}{2}\frac{1}{2}} \otimes (D^{\frac{1}{2}0} \oplus D^{0\frac{1}{2}}) = (D^{\frac{1}{2}0} \oplus D^{0\frac{1}{2}}) \oplus (D^{1\frac{1}{2}} \oplus D^{\frac{1}{2}1})$. Because $(D^{1\frac{1}{2}} \oplus D^{\frac{1}{2}1})(R) = \mathscr{D}^{\frac{3}{2}} \oplus \mathscr{D}^{\frac{3}{2}} \oplus \mathscr{D}^{\frac{1}{2}} \oplus \mathscr{D}^{\frac{1}{2}}$ and $(D^{\frac{1}{2}0} \oplus D^{0\frac{1}{2}})(R) = \mathscr{D}^{\frac{1}{2}} \oplus \mathscr{D}^{\frac{1}{2}}$, $\boldsymbol{X}_p = \boldsymbol{X}^{\frac{3}{2}}_p \oplus \boldsymbol{X}^{\frac{1}{2}}_p$ where $\dim \boldsymbol{X}^{\frac{3}{2}}_p = 4$, $\dim \boldsymbol{X}^{\frac{1}{2}}_p = 16$. From the definition of the parity-invariant subalgebra $\boldsymbol{X}^{\pi}_p$ we have $\boldsymbol{X}^{\pi}_p = \boldsymbol{X}^{\pi\frac{3}{2}}_{p+} \oplus \boldsymbol{X}^{\pi\frac{3}{2}}_{p-} \oplus \boldsymbol{X}^{\pi\frac{1}{2}}_{p+} \oplus \boldsymbol{X}^{\pi\frac{1}{2}}_{p-}$ where $\dim \boldsymbol{X}^{\pi\frac{3}{2}}_{p\pm} = 1$, $\dim \boldsymbol{X}^{\pi\frac{1}{2}}_{p\pm} = 4$. According to the discussion in § 2, the subalgebras $\boldsymbol{X}^{\pi\frac{3}{2}}_{p\pm}$ and $\boldsymbol{X}^{\pi\frac{1}{2}}_{p\pm}$ have the bases $\{X^{\frac{3}{2}}_{[1,\frac{1}{2}],[1,\frac{1}{2}]\pm}\}$ and $\{X^{\frac{1}{2}}_{[1,\frac{1}{2}],[1,\frac{1}{2}]\pm},\ X^{\frac{1}{2}}_{[1,\frac{1}{2}],[\frac{1}{2},0]\pm},\ X^{\frac{1}{2}}_{[\frac{1}{2},0],[1,\frac{1}{2}]\pm},\ X^{\frac{1}{2}}_{[\frac{1}{2},0],[\frac{1}{2},0]\pm}\}$ respectively. Similarly, the two-dimensional subalgebra $\boldsymbol{Y}^{\pi} = \boldsymbol{Y}^{\pi(1,\frac{1}{2})} \oplus \boldsymbol{Y}^{\pi(\frac{1}{2},0)}$ is spanned by $Y^{\pi(1,\frac{1}{2})} = X^{\frac{3}{2}}_{[1,\frac{1}{2}],[1,\frac{1}{2}]+} + X^{\frac{3}{2}}_{[1,\frac{1}{2}],[1,\frac{1}{2}]-} + X^{\frac{1}{2}}_{[1,\frac{1}{2}],[1,\frac{1}{2}]+} + X^{\frac{1}{2}}_{[1,\frac{1}{2}],[1,\frac{1}{2}]-}$ and $Y^{\pi(\frac{1}{2},0)} = X^{\frac{1}{2}}_{[\frac{1}{2},0],[\frac{1}{2},0]+} + X^{\frac{1}{2}}_{[\frac{1}{2},0],[\frac{1}{2},0]-}$. Because in our case the vector–spinor field $\psi_{\mu\lambda}$, $\mu = 0, 1, 2, 3$, $\lambda = 1, 2, \dot{1}, \dot{2}$, describes the spin-$\frac{3}{2}$ particles, the projector $\Pi = \Pi^{\frac{3}{2}} = X^{\frac{3}{2}}_{[1,\frac{1}{2}],[1,\frac{1}{2}]+} + X^{\frac{3}{2}}_{[1,\frac{1}{2}],[1,\frac{1}{2}]-}$ while $(I - \Pi) = \Pi^{\frac{1}{2}}$. Consequently equation (40*b*) is satisfied trivially. From equations (19)–(21) and (23) we obtain immediately that the Hermitian matrices representing $L^{s}_{1\pm}/\sqrt{p^2}$ and $L^{s}_{0\pm}$ have the form

$$\frac{L^{\frac{3}{2}}_{1\pm}}{\sqrt{p^2}} = \pm 4\lambda_{1[1,\frac{1}{2}],[1,\frac{1}{2}]}, \qquad L^{\frac{3}{2}}_{0\pm} = v^{(1,\frac{1}{2})},$$

$$\frac{L^{\frac{1}{2}}_{1\pm}}{\sqrt{p^2}} = \begin{pmatrix} \pm 2\lambda_{1[1,\frac{1}{2}],[1,\frac{1}{2}]} & \sqrt{3}\,\lambda_{1[1,\frac{1}{2}],[\frac{1}{2},0]} \\ \sqrt{3}\,\lambda^{*}_{1[1,\frac{1}{2}],[\frac{1}{2},0]} & \pm 2\lambda_{1[\frac{1}{2},0],[\frac{1}{2},0]} \end{pmatrix}, \qquad L^{\frac{1}{2}}_{0\pm} = \begin{pmatrix} v^{(1,\frac{1}{2})} & 0 \\ 0 & v^{(\frac{1}{2},0)} \end{pmatrix},$$

where the coefficients $v^{(AB)}$ are different from zero. From the condition (40*a*) the matrices $[(L^{\frac{1}{2}}_{0\pm})^{-1}(L^{\frac{1}{2}}_{1\pm}/\sqrt{p^2})]$ are nilpotent. Thus it is trivial to show that

$$|\lambda_{1[1,\frac{1}{2}],[\frac{1}{2},0]}| = (2/\sqrt{3})(\lambda_{1[1,\frac{1}{2}],[1,\frac{1}{2}]}\lambda_{1[\frac{1}{2},0],[\frac{1}{2},0]})^{\frac{1}{2}},$$

$$v^{(\frac{1}{2},0)} = -v^{(1,\frac{1}{2})}(\lambda_{1[\frac{1}{2},0],[\frac{1}{2},0]}/\lambda_{1[1,\frac{1}{2}],[1,\frac{1}{2}]}).$$

Denoting

$$\phi = \frac{\sqrt{3}\,\lambda_{1[1,\frac{1}{2}],[\frac{1}{2},0]}}{4\lambda_{1[1,\frac{1}{2}],[1,\frac{1}{2}]}}, \qquad m = \frac{v^{(1,\frac{1}{2})}}{4\lambda_{1[1,\frac{1}{2}],[1,\frac{1}{2}]}}, \qquad c = \lambda_{1[1,\frac{1}{2}],[1,\frac{1}{2}]},$$

we obtain the operator $L$ in the final form

$$\begin{aligned} c^{-1}L = \sqrt{p^2}[&(X^{\frac{3}{2}}_{[1,\frac{1}{2}],[1,\frac{1}{2}]+} - X^{\frac{3}{2}}_{[1,\frac{1}{2}],[1,\frac{1}{2}]-}) + \tfrac{1}{2}(X^{\frac{1}{2}}_{[1,\frac{1}{2}],[1,\frac{1}{2}]+} - X^{\frac{1}{2}}_{[1,\frac{1}{2}],[1,\frac{1}{2}]-}) \\ &+ \phi(X^{\frac{1}{2}}_{[1,\frac{1}{2}],[\frac{1}{2},0]+} + X^{\frac{1}{2}}_{[1,\frac{1}{2}],[\frac{1}{2},0]-}) + \phi^*(X^{\frac{1}{2}}_{[\frac{1}{2},0],[1,\frac{1}{2}]+} + X^{\frac{1}{2}}_{[\frac{1}{2},0],[1,\frac{1}{2}]-}) \\ &+ 2|\phi|^2(X^{\frac{1}{2}}_{[\frac{1}{2},0],[\frac{1}{2},0]+} - X^{\frac{1}{2}}_{[\frac{1}{2},0],[\frac{1}{2},0]-})] + m(Y^{\pi(1,\frac{1}{2})} - 4|\phi|^2 Y^{\pi(\frac{1}{2},0)}). \end{aligned}$$

Using the explicit formulae for the base elements $X$ and $Y$, given in appendix 3, we rewrite the Rarita–Schwinger equation in the standard form

$$\begin{aligned} [\![ [p\gamma\delta^{\nu}_{\mu} + \tfrac{1}{4}(\tfrac{3}{2} + (\phi + \phi^*)/\sqrt{3} + 2|\phi|^2)\gamma_\mu p\gamma\gamma^\nu - \tfrac{1}{2}(1 + 2\phi/\sqrt{3})p_\mu\gamma^\nu - \tfrac{1}{2}[1 + (2\phi^*/\sqrt{3})]\gamma_\mu p^\nu] \\ + m[\delta^{\nu}_{\mu} - (\tfrac{1}{4} + |\phi|^2)\gamma_\mu\gamma^\nu]]\!]\psi_\nu(p) = 0. \end{aligned}$$

Here $g_{\mu\nu} = \mathrm{diag}(+---)$, $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. From the formulae (34) and (35) we can obtain the operators $M(p)$ and $D(p)$.

## 7. Final remarks

Let us summarise the results. We have introduced the description of the free field equations (satisfying the reasonable conditions A–D of § 1) using the notion of the commutant $\boldsymbol{X}_p$. This description is free from the explicit $p_\mu(\partial_\mu)$ dependence and convenient for the formulation of algebraic conditions like B, C, D of § 1. The general form of the free field equations has been given in this framework (equations (16), (17*a*), (19), (21*a*, *b*), (22*a*–*d*), (23)) and the existence conditions formulated (equations (28*a*, *b*), (29*a*, *b*), (32), (33), (36), (37)). For unique spin theories these conditions give very stringent limitations on the possible form of the free field equations†. In § 2 the explicit form of the base elements of the commutant $\boldsymbol{X}_p$ for an arbitrary representation of the Lorentz group is given.

Summarising, the problem of constructing of the free field equations is reduced to finding the matrices satisfying the algebraic formulae (28) and (29), or, in the cases $L_2 = 0$ or $L_1 = 0$, to finding the nilpotent matrices (see equation (40*a*)) or matrices with a power orthogonal to the physical spin space (see equation (40*b*)).

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## Appendix 1. Weyl's theorem (Weyl 1939)

Let D be a completely reducible representation of the group $G$ in the vector space $V = \oplus V_k$ where $V_k$ are irreducible subspaces. Then the commutant $\boldsymbol{X}$ of $D$, defined by $[D, \boldsymbol{X}] = 0$, forms the associative algebra spanned by the operators $X_{ik}$ which intertwine

† If, for example, the matrices $L_k/(\sqrt{p^2})^k$ and $L_0$ are simultaneously diagonalisable, then the unique-spin equations exist only for $s = 0, \frac{1}{2}, 1$. If $\pm L_{k\pm}/(\sqrt{p^2})^k$ $(k = 1, 2)$ are non-negative definite, then the unique-spin equations exist only for $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$.

the irreducible subspaces $V_k$ and $V_i$ (and irreducible equivalent representations of $G$ acting in $V_k$ and $V_i$). The operators $X_{ik}$ are determined uniquely, and $X_{ij}X_{lk} = \delta_{jl}X_{ik}$ if $i, j, l, k$ denote the equivalent representations, while $X_{rs} = 0$ otherwise.

Note that $X_{kk}$ are the projection operators.

## Appendix 2. Derivative of the formulae (28)–(29) and (30)–(31)

For simplicity we restrict ourselves to the derivation of the simplified version (32)–(35) of equations (28)–(31), namely to the case $L_2 = 0$. The general formulae can be obtained quite analogously.

If we expand the operator $M(p)$ in the form $M(p) = \Sigma_{k=-m}^{n} (\sqrt{p^2})^k M_k(p)$, where the $M_k$ have the homogeneity degree with respect to $p_\mu$ equal to zero, then from equations (16) and (26) we obtain

$$(\sqrt{p^2})^{n+1}(M_n L_1/\sqrt{p^2}) + \sum_{k=-m}^{n-1} (\sqrt{p^2})^{k+1}[M_k L_1/\sqrt{p^2} + M_{k+1}L_0] + (\sqrt{p^2})^{-m} M_{-m}L_0 = I - \Pi.$$

Note that the homogeneity degree of the products $M_k L_1/\sqrt{p^2}$ and $M_k L_0$ is equal to zero, while for $(\sqrt{p^2})^k$ it is equal to $k$. On the other hand, the homogeneity degree of $I - \Pi$ is equal to zero. Therefore the above equation is equivalent to the system

$$\begin{aligned}
&M_n L_1/\sqrt{p^2} = 0,\\
&M_k L_1/\sqrt{p^2} + M_{k+1}L_0 = 0 \qquad \text{for } k \geqslant 0 \text{ and } -m \leqslant k \leqslant -2,\\
&M_{-1}L_1/\sqrt{p^2} + M_0 L_0 = I - \Pi,\\
&M_{-m}L_0 = 0.
\end{aligned}$$

Because of the invertibility of $L_0$ we obtain

$$\begin{aligned}
&M_{-m} = M_{-m+1} = \ldots = M_{-1} = 0,\\
&(I - \Pi)(L_0^{-1}L_1)^{n+1} = 0,\\
&M(p) = (I - \Pi)\Big(\sum_{k=0}^{n} (-L_0^{-1}L_1)^k\Big)L_0^{-1},
\end{aligned}$$

i.e. equations (32) and (34) hold. Applying the same considerations to $D(p)$, we obtain the equations (33) and (35).

## Appendix 3. Basis of the commutant $X_p^\pi$ for the representation $D^{\frac{1}{2}\frac{1}{2}} \otimes (D^{\frac{1}{2}0} \oplus D^{0\frac{1}{2}}) = (D^{\frac{1}{2}0} \oplus D^{0\frac{1}{2}}) \oplus (D^{1\frac{1}{2}} \oplus D^{\frac{1}{2}1})$ $([(AB)\alpha] \equiv [AB])$

$$\begin{aligned}
X^{\frac{1}{2}}_{[\frac{1}{2},0],[\frac{1}{2},0]\pm\mu}{}^{\nu} &= \tfrac{1}{8}[\gamma_\mu\gamma^\nu \pm (1/\sqrt{p^2})\gamma_\mu p\gamma\gamma^\nu],\\
X^{\frac{1}{2}}_{[1,\frac{1}{2}],[\frac{1}{2},0]\pm\dot{\mu}}{}^{\nu} &= \pm(1/4\sqrt{3})\{[\tfrac{1}{2}\gamma_\mu\gamma^\nu - (2/p^2)p\gamma p_\mu\gamma^\nu] \mp (1/\sqrt{p^2})[p_\mu\gamma^\nu + \tfrac{1}{2}p\gamma\gamma_\mu\gamma^\nu]\},\\
X^{\frac{1}{2}}_{[\frac{1}{2},0],[1,\frac{1}{2}]\pm\mu}{}^{\nu} &= \pm(1/4\sqrt{3})\{[\tfrac{1}{2}\gamma_\mu\gamma^\nu - (4/p^2)p_\mu p^\nu + (2/p^2)p\gamma\gamma_\mu p^\nu]\\
&\quad \mp (1/\sqrt{p^2})[2\gamma_\mu p^\nu - p_\mu\gamma^\nu + \tfrac{1}{2}p\gamma\gamma_\mu\gamma^\nu]\},\\
X^{\frac{3}{2}}_{[1,\frac{1}{2}],[1,\frac{1}{2}]\pm\mu}{}^{\nu} &= \tfrac{1}{2}(1 \pm p\gamma/\sqrt{p^2})\{\delta_\mu{}^\nu - \tfrac{1}{3}[\gamma_\mu\gamma^\nu + (2/p^2)p_\mu p^\nu - (1/p^2)p\gamma(p_\mu\gamma^\nu - \gamma_\mu p^\nu)]\},
\end{aligned}$$

$$X^{\frac{1}{2}}_{[1,\frac{1}{2}],[1,\frac{1}{2}]\pm\mu}{}^{\nu} = \tfrac{1}{12}\{[\tfrac{1}{2}\gamma_\mu\gamma^\nu + (4/p^2)p_\mu p^\nu - (2/p^2)p\gamma(p_\mu\gamma^\nu - \gamma_\mu p^\nu)]$$
$$\pm (1/\sqrt{p^2})[(8/p^2)p\gamma p_\mu p^\nu - 2\gamma_\mu p^\nu - p_\mu\gamma^\nu - \tfrac{1}{2}p\gamma\gamma_\mu\gamma^\nu]\}.$$

The projector on the spin-$\frac{3}{2}$ subspace is given by

$$\Pi^{\frac{3}{2}}{}_{\mu}{}^{\nu} = \delta_\mu{}^\nu - \tfrac{1}{3}[\gamma_\mu\gamma^\nu + (2/p^2)p_\mu p^\nu - (1/p^2)p\gamma(p_\mu\gamma^\nu - \gamma_\mu p^\nu)]$$

while the base elements of $\boldsymbol{Y}^\pi$ (in this case the projectors on the representations $D^{\frac{1}{2}0}\oplus D^{0\frac{1}{2}}$ and $D^{\frac{1}{2}0}\oplus D^{0\frac{1}{2}}$) are

$$Y^{\pi(\frac{1}{2},0)}{}_{\mu}{}^{\nu} = \tfrac{1}{4}\gamma_\mu\gamma^\nu, \qquad Y^{\pi(1,\frac{1}{2})}{}_{\mu}{}^{\nu} = \delta_\mu{}^\nu - \tfrac{1}{4}\gamma_\mu\gamma^\nu.$$

The base elements spanning the commutant $\boldsymbol{X}_p$ can be obtained from the base vectors of $\boldsymbol{X}_p^\pi$ by the action of the projectors $\frac{1}{2}(1 \pm \gamma_5)$.

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